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Lanchester Law Model with Fractional Distribution of Fire and Variable Performance Against Multiple Opponents

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Abstract

Lanchester's law serves as a deterministic mathematical model of a battle between two or more opponents. We build a new model that incorporates both variable offense and variable defense, as well as the commander's ability to focus fire on a selected enemy. We show that in a situation where an army fights two allies, the correct strategy of focusing on the weaker enemy leads to an eventual win, whereas the incorrect decision to split the fire equally leads to the army's annihilation.

KEY WORDS: *Lanchester law, differential equations, battle simulation, command and control, military model*

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1. Introduction

It is an unavoidable fact that war brings suffering to the people. War never changes in this aspect. War causes harm to the civilians who face adversities such as food and product scarcity, military drafts, collateral damage, or direct military attacks. However, the war is undeniably worse for soldiers who face the enemy directly on the battlefield. There are many adverse consequences of military action for soldiers. Among them, there are direct physical consequences such as death or body harm, mental consequences such as post-traumatic stress disorder, or social consequences such as feeling disconnected from civilian life [1].

The significant cost of war or even an individual battle is then apparent. In many scenarios, it would be best for ordinary civilians to stay out of the conflict entirely. However, it is not always possible to avoid fighting. Therefore, it is essential to learn from past conflicts and identify and study past mistakes to lessen the suffering of soldiers. Furthermore, it is essential to develop the ability to analyze future conflicts virtually, without the actual battle and its associated suffering. Both aforementioned tasks can be accomplished with mathematical models [2]. Deterministic models form an important branch of mathematical models as they describe a predictable system that produces fixed outputs for given inputs. Among deterministic models, the so-called mechanistic models are built based on fundamental principles such as physical, biological, economic, and other laws. See, for example, the well-known Hodgkin-Huxley model of neuron signal transmission [3] and many models that followed it. These models often employ differential equations to model the interactions between the system's variables. This is also true in this article, where we utilize a Lanchester law to develop a new generalization of a mechanistic model, which consists of a system of linear differential equations.

Lanchester's law mathematically describes the strengths of opposing military forces in a direct confrontation. Hence, the name Lanchester combat model is also sometimes used. Lanchester's combat model was developed around the time of the First World War, and the standard model has been generalized in many directions since then. The Lanchester combat model with unaimed fire was investigated, for example, in [4]**.** A combat model for spatially distributed armies was studied in [6]**.** Random stochastic effects were incorporated in [7–9] to derive the so-called stochastic Lanchester combat model.

In this article, we follow in the footsteps of the article [4]. There, the authors developed a multi-battle warfare model that simulates conflicts between multiple combat forces. Such conflicts are not uncommon. In many conflict there are multiple allies on both sides, and if they meet on a battlefield, such as at the Battle of Austerlitz, the multi-battle warfare model can be employed. However, the multi-battle warfare model assumes that if an army faces two enemies, it splits its fire equally to fight both enemies simultaneously. The article [4] focuses primarily on an unaimed fire, and such an assumption then makes sense. However, this assumption is unrealistic when aimed fire is considered instead. To mitigate this drawback, we have developed a new generalized model that enables the fractional distribution of fire. In other words, our model enables the commander to focus fire on one of the opponents and tackle the other once the first one is eliminated [5]. To validate our model, we present a scenario in which the equal distribution of forces results in a loss, but focused fire leads to a win.

Additionally, the multi-battle warfare model assumes variable offense capability of military forces but a fixed defense capability. Our model, on the other hand, works with both variable offense and defense capability. This enables the analysis of scenarios where a military force performs better against one of its opponents. Additionally, it enhances the model's applicability.

2. Literature review about Lanchester's law

During World War I, the first papers developing differential equations to model the dynamics of a battle between two combatants (nowadays known as Lanchester's laws) had occurred. Namely, the paper [10] by M. Osipov and [11] by F. W. Lanchester appeared. In the course of time and by the situation of World War II, many mathematicians, scientists, and engineers were inspired by these combat laws and published papers researching related issues. For instance, P. M. Morse and G. E. Kimball [12] amplified the multiple-element engagement in Lanchester Square law in [12]. In 1962, S.J. Deitchman generalized the Lanchester model to the (asymmetric) guerrilla model [13], which was improved by M.B. Schaffer in 1968 by involving the discipline and morale of the troops into the model [14]. At the beginning of the new millennium, E.H. Kaplan et al. and M. Kress and R. Szechtmann extended the asymmetric Lanchester model and described counter-terrorism models (see [15,16], respectively; for another counter-terrorism model, we refer, e.g., to [17]. For a summarization of combat modelling in a broader aspect, we refer to, e.g., [18,19]. Another (and very important) generalization of Lanchester's laws consisted of studying combat models with more than two parties involved in the battle; we refer, e.g., to [4,20–24].

Only recently, a so-called network-centric warfare, emphasizing the role of battle information more than in classical warfare, became a point of interest for many scientists and mathematicians [25,26]. However, incorporating the impact of received information during a battle into the combat model is a non-trivial task, requiring the analysis of nonlinear Lanchester equations [27, 28]. The incorporation of random processes into the coefficients of Lanchester equations is another research topic of mainly the past few years in this area (see, e.g., [29] where the stochastic combat models are developed under the theory of Markov processes, [30], or [8] where the Gaussian and uniformly distributed random parameters of Lanchester equations are considered). Nevertheless, some works considering randomness in the studied combat model can be found in the past millennium as well in [31] where a Monte Carlo method is used, or [7,32].

Among the papers referred to above, there are numerous other works related to this topic. From these, we enhanced, e.g., the papers [33, 34], which demonstrate the validity of the (linear) Lanchester model using historical data from real combat situations. For the discrete analogue of the Lanchester model, we refer, e.g., to [35]. In [36], optimal control theory is applied to achieve optimal fire distributions across several Lanchester-type models. The optimal policy was discussed in [37] as well, but for a one-against-many combat. Finally, we refer to the papers [38] and [6], which introduce the extension of the Lanchester model to spatial dimensions by incorporating fractal effects (such as the spatial distribution of armies, congestion, and finite engagement ranges).

3. Multi-battle model with fractional distribution of fire

In this section we describe multi-battle warfare model defined in [4] and describe its generalization in detail the new multi-battle model with fractional distribution of fire. From here onwards we denote as $y_i(t)$ the size of the army i, $i \in \{1, \dots, n\}$, at the time t. Emphasize that $t \geq 0$ in all our considered models, i.e., $y_i(0)$ is the initial size of the army i. To simplify the notation let y_i signify the army i.

The derivative $y_i'(t)$ represents the rate of change of the size of army i, $i \in \{1, \dots, n\}$, at the time t. The column vector $y(t) = (y_1(t), \dots, y_n(t))^T$ represents the size of all the armies participating in the battle. The vector of derivatives $y'(t) = (y_1'(t), \dots, y_2'(t))^T$ represents the rate of change for alle the participating armies. Variable o_i describes the offence capability of army i, $o_i > 0$, $i \in \{1, \dots, n\}$ (detailed description of o_i is below). Matrix $O = (o_{i,j})_{i,j=1,..,n}$ is diagonal matrix such that

$$o_{i,j} = \begin{cases} o_i, & i = j, \\ 0, & i \neq j. \end{cases} \tag{1}$$

The multi-battle warfare model (for more details see [4]) is then given as

$$y' = -COy, \tag{2}$$

where $y = (y_1, \dots, y_n)^T$, $y' = (y_1', \dots, y_n')^T$ and matrix $C = (c_{i,j})_{i,j=1,..,n}$ is the weighted conflict matrix such that

$$c_{i,j} = \frac{\chi(i,j)}{\sum_{k=1}^{n} \chi(k,j)}. \tag{3}$$

In (3) $\chi(i,j)$ is a function that is equal to one when armies y_i and y_j fight against each other and zero otherwise. The denominator of coefficients $c_{i,j}$ represent equal distribution of forces among all the opponents. For example, if an army y_2 faces two opponents, then $c_{i,2} = \frac{1}{2}$ for each opponent y_i and if it faces five opponents then $c_{i,2} = \frac{1}{5}$ for each opponent y_i. Additionally, $c_{i,i} = 0$ for all i because otherwise it would mean that the army y_i fights against itself. It is also assumed that the conflict is bilateral, in other words, that if army y_i fights army y_j (i.e. $\chi(i,j) = 1$) then also army y_j fights army y_i (i.e. $\chi(j,i) = 1$). Hence, the matrix C is symmetric.

Multi-battle model with fractional distribution of fire (MBMFD) is given in the following way

$$y' = -D\bar{C}Oy, \tag{4}$$

such that y, y' and O have the same interpretation as with the multi-battle warfare model in (2). However, matrix $\bar{C} = (\bar{c}_{i,j})_{i,j=1,\dots,n}$ is the fractional conflict matrix such that

$$\bar{c}_{i,j} = \chi(i,j)p(i,j). \tag{5}$$

Function $\chi(i,j)$ is again a function that is equal to one when armies y_i and y_j fight against each other and zero otherwise. Define the portion of fire $p(i,j)$, $i,j \in \{1,\dots,n\}$, that the army j aims at the army i such that $p(i,j) \in [0,1]$, $p(i,i) = 0$ and

$$\sum_{k=1}^{n} p(k,j) = 1. \tag{6}$$

For example, $p(i,j) = 0.2$ means that the army j sends 20% of $y_j(0)$ against the army i and the rest against some other armies. Variable d_i describes the defence capability of army i, $d_i > 0$, $i \in \{1,\dots,n\}$ (detailed description of d_i is below). Finally, the matrix $D = (d_{i,j})_{i,j=1,\dots,n}$ is a diagonal matrix such that

$$d_{i,j} = \begin{cases} d_i, & i = j, \\ 0, & i \neq j. \end{cases} \tag{7}$$

For example, if $n = 3$, then system (4) takes the form

$$\begin{aligned} y_1' &= -d_1[\chi(1,2)p(1,2)o_2y_2 + \chi(1,3)p(1,3)o_3y_3], \\ y_2' &= -d_2[\chi(2,1)p(2,1)o_1y_1 + \chi(2,3)p(2,3)o_3y_3], \\ y_3' &= -d_3[\chi(3,1)p(3,1)o_1y_1 + \chi(3,2)p(3,2)o_2y_2]. \end{aligned} \tag{8}$$

Observe that the constants o_1 and d_1 are all associated with the army y_1. These constants characterize its offence capabilities and defense capabilities. Further the constants $p(2,1)$ and $p(3,1)$ characterize division of forces of y_1 against other armies; note that $p(2,1) + p(3,1) = 1$. Analogous statements apply to the remaining two armies. Finally, the numbers $\chi(i,j)$, $i,j \in \{1,2,3\}$, set which army fights which.

Offence capability o_i of the army y_i represents the ability of the soldiers in the army y_i to eliminate opposing soldiers. Higher o_i then means more eliminated opponents which might be the result of a better weaponry, better training with weapons, better tactics, etc. On the other hand, defence capability d_i represents the ability of the soldiers in the army y_i to survive the enemy fire due to better protective equipment, better cover, better training or due to other reasons. Lower d_i means better ability to survive.

Now, we set $\chi(1,2) = 1$, $\chi(1,3) = 1$, $\chi(2,3) = 0$, $p(1,2) = 1$, $p(1,3) = 1$, $d_1 = 1$ and $o_2 = o_3 = 1$ and (8) takes the form

$$\begin{aligned} y_1' &= -y_2 - y_3, \\ y_2' &= -d_2p(2,1)o_1y_1, \\ y_3' &= -d_3p(3,1)o_1y_1. \end{aligned} \tag{9}$$

Here the first army y_1 fights against two allies y_2 and y_3. The armies y_2 and y_3 has equal (unit) offence capabilities and the army y_1 has unit defense capability. In addition, if we set $p(2,1) = 0.8$, then $p(3,1) = 0.2$, i.e., 80% of $y_1(0)$ are sent against the army 2 and 20% against army 3. We will examine this specific system in more detail in the next section.

The MBMFD is more complex than the original multi-battle model because it introduces n^2 new parameters. However, it is also more realistic because it allows the incorporation of elementary tactics such that the commander decides against whom to direct the fire. The MBMFD also retains one drawback of multi-battle warfare model and that is that $y_i(t)$ can be eventually negative. Hence, MBMFD is then used until $y_i(t) > 0$ for all i and if at any point t_0 exist i such that $y_i(t_0) = 0$ then a submodel must be used from then on. For $t > t_0$ let us consider the submodel

$$\tilde{y}' = -D\overline{C_2}O\tilde{y}, \tag{10}$$

where $\tilde{y} = (y_1,\dots,y_{i-1},0,y_{i+1},\dots,y_n)^T$ and $\tilde{y}' = (y_1',\dots,y_{i-1}',0,y_{i+1}',\dots,y_n')^T$ as we replace i-th coefficient with a zero and modified matrix $\overline{C_2}$ reflects the fact that the army i was eliminated and the distribution of fire has to be changed. Note that $y_j(t_0)$ for $j \in \{1,\dots,n\}$, $i \neq j$, are the initial sizes of armies in the new conflict modeled by (10) for $t > t_0$. However, this is necessary only when the battle continues. When $y_i(t) = 0$ it might also mean that one side lost and that the battle is over.

4. Impact of a proper distribution of fire

In this section, we provide an example of a battle where two different distributions of fire lead to two different outcomes of the battle. Let us return to system (9), i.e., the system where there are three armies and the first army y_1 fights against two allies y_2 and y_3. Both sides have equal number of soldiers, in other words, $y_1(0) = y_2(0) + y_3(0)$; to simplify the calculations let us assume $y_2(0) = y_3(0)$. All the armies have the same offence capabilities, i.e., in (9) we set $o_1 = 1$ as $o_2 = o_3 = 1$. On the other hand, defensive capabilities are different, we set $d_2 = 1.2$ and $d_3 = 0.8$. Recall $d_1 = 1$. Then the model (9) becomes

$$\begin{aligned} y_1' &= -y_2 - y_3, \\ y_2' &= -1.2p(2,1)y_1, \\ y_3' &= -0.8p(3,1)y_1. \end{aligned} \tag{11}$$

Army y_1 has three main options on how to distribute the fire. It can split the fire equally; it can focus fire on the army y_2; or it can focus fire on the army y_3. The inequality $d_2 > d_3$ means that the soldiers in the army y_3 have a better ability to survive than the soldiers in the army y_2. Because both armies y_2 and y_3 have initially the same number of soldiers, i.e. $y_2(0) = y_3(0)$, it is reasonable for the army y_1 to initially focus fire on y_2 until it is eliminated.

To analyze system (11), let us consider $(y_1(0), y_2(0), y_3(0))$ as a general initial state. Then system (11) has the solution of the form

$$\begin{aligned} y_1(t) &= y_1(0)\cosh\lambda t - \tfrac{y_2(0)+y_3(0)}{\lambda}\sinh\lambda t, \\ y_2(t) &= \tfrac{1.2p(2,1)}{\lambda}\left[\tfrac{y_2(0)+y_3(0)}{\lambda}\cosh\lambda t - y_1(0)\sinh\lambda t\right] + \tfrac{0.8p(3,1)y_2(0)-1.2p(2,1)y_3(0)}{\lambda^2}, \\ y_3(t) &= \tfrac{0.8p(3,1)}{\lambda}\left[\tfrac{y_2(0)+y_3(0)}{\lambda}\cosh\lambda t - y_1(0)\sinh\lambda t\right] - \tfrac{0.8p(3,1)y_2(0)-1.2p(2,1)y_3(0)}{\lambda^2}, \end{aligned} \tag{12}$$

where $\lambda = \sqrt{1.2p(2,1) + 0.8p(3,1)}$. The solution can be computed even in the case of the more general system (9), since it is a system of linear differential equations with constant coefficients, see [39]. Indeed, generally, the individual component $y_i(t)$ of the solution has form $\sum_{k=1}^{r} p_k(t)\exp(\lambda_k t)$, where $\lambda_1, \lambda_2, \ldots, \lambda_r$ are eigenvalues (generally complex) of the matrix of the system (for (4) it is the matrix $-D\bar{C}O$) and $p_1, p_2, \ldots, p_r$ are suitable polynomials such that degree of p_k is less than multiplicity of λ_k as a root of the so-called characteristic equation associated with the matrix of the system. Further recall that $\cosh t = [\exp(t) + \exp(-t)]/2$ and $\sinh t = [\exp(t) - \exp(-t)]/2$. Hence, it is easily seen that solution (12) has the correct form and by substituting it into system (11), it can be verified that it is indeed a solution.

Since we have exact algebraic solution (12), we can analyze its components using graphical visualization (no need to use numerical methods for plotting) and by direct analysis of algebraic expressions. Note that it follows directly from the equations of system (11) that all components of the solution are decreasing as long as all three solutions are positive. Indeed, in that case, the right-hand side of each equation is negative, hence, y_1', y_2', y_3' are negative.

First, let us focus on (11) with $p(2,1) = 0.5$ and $p(3,1) = 1 - p(2,1) = 0.5$. It means that the army y_1 equally divides their forces to fight against y_2 and y_3. To satisfy the conditions $y_1(0) = y_2(0) + y_3(0)$ and $y_2(0) = y_3(0)$, let us set $y_1(0) = 1$ and $y_2(0) = 0.5 = y_3(0)$. Recall that this means that the sides are equal in terms of the size of their armies. In Fig. 1, we can see the graphs of the solution for $t \in [0, t_0]$, where t_0 is the time, when some of the armies vanish and $y_i(t) > 0$ for $t \in [0, t_0)$ and $i \in \{1,2,3\}$.

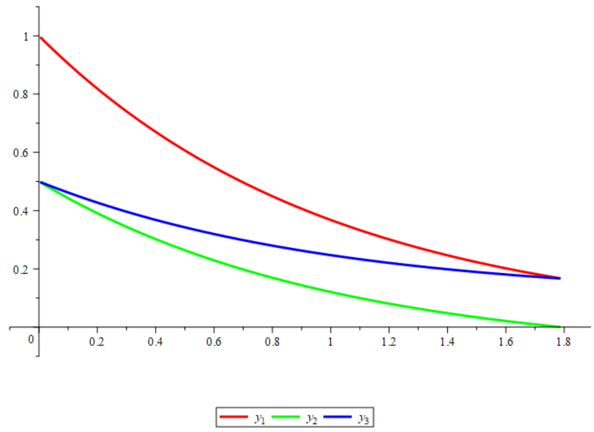

Fig. 1. The case $p(2,1) = 0.5$. Here t_0 is approximately 1.79. It is zero point of the army y_2

In t_0, the sizes of the opposing armies y_1 and y_2 are still positive, in fact, they are equal. Therefore, in order to specify the winner, we consider the submodel

$$\begin{aligned} y_1' &= -y_3, \\ y_3' &= -0.8y_1. \end{aligned} \tag{13}$$

Emphasize formal changes compared to (11) are $y_2 = 0$, $y_2' = 0$ and $p(2,1) = 0$ and therefore $p(3,1) = 1$, but we do not actually consider system (13) to be a special case of (11). The pair $(y_1(t_0), y_3(t_0))$ is the initial state for (13). Then again, the particular solution to the corresponding initial problem can be expressed algebraically. However, we will only express the solution graphically here. In Fig. 2, we can see the components of solution of (13) plotted on the interval $[t_0, t_1]$, where t_1 is the time, when one of the armies vanish and $y_i(t) > 0$ for $t \in [t_0, t_1)$ and $i \in \{1,3\}$.

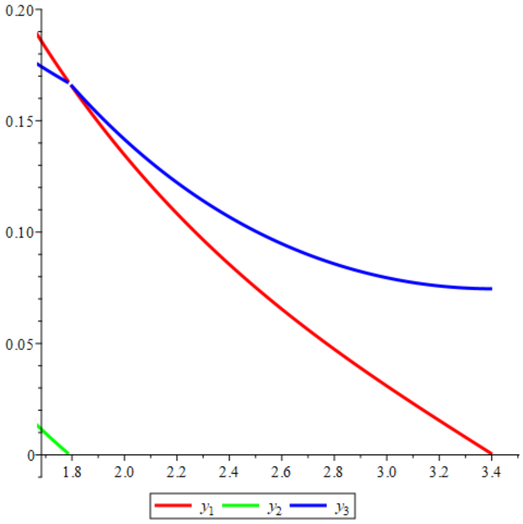

Fig. 2. The submodel for the case $p(2,1) = 0.5$. Here t_1 is approximately 3.41. It is zero point of the army y_1.

For clarity, we have not set the same scale on the axes in Fig. 2, so the slope of the graphs does not correspond to Fig. 1, and we have also shown some of the graphs for the solutions of (11). Although the colors of the lines match, it is important to realize that these are different functions, which are, however, continuously connected at point t_0 and correspond to the same army.

From Fig. 1 and Fig. 2, the result of the battle is clear. The alliance of y_2 and y_3 win, but y_3 was destroyed, while y_2 survived with approximately 15% of initial size $y_2(0)$.

Let us move to the case (11) with $p(2,1) = 0.8$ and $p(3,1) = 0.2$. The initial sizes of the armies will be the same as in the previous case, i.e., $y_1(0) = 1$ and $y_2(0) = 0.5 = y_3(0)$. The components of the corresponding solution are shown in Fig. 3 on the left-hand side; on the right-hand side there are components of a solution for the submodel that is the same as in the previous case. From Fig. 3 it is clearly seen that the larger army y_1 wins, since both y_2 and y_3 are eventually destroyed. In fact, we prove that if we fix all parameters except $p(2,1)$ in (11), then there is such a parameter setting (with the conditions $y_1(0) = y_2(0) + y_3(0)$ and $y_2(0) = y_3(0)$) that the battle can end in both ways according to the setting of parameter $p(2,1)$. Of course, this statement is very specific, but it suggests to us that there are probably certain classes of cases in which the system behaves qualitatively similarly according to some criterion. The search for such classes may be the subject of further investigation.

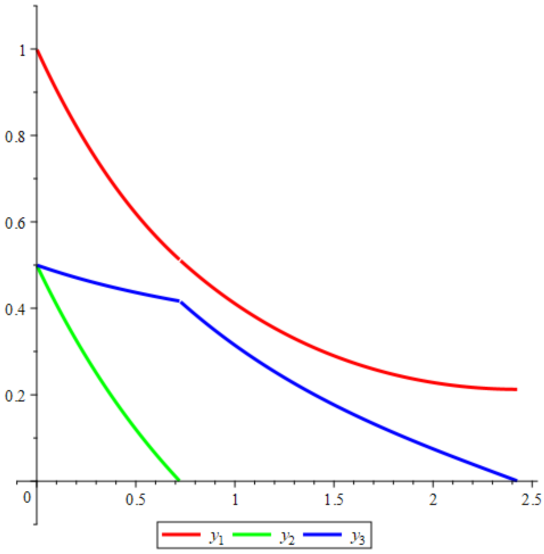

Fig. 3. The case $p(2,1) = 0.8$ (with submodel). Here t_0 is approximately 0.72 and t_1 is approximately 2.43.

5. Lanchester Square Law

In this section, we derive Lanchester square law for the system

$$\begin{aligned} y_1' &= -d_1[o_2y_2 + o_3y_3], \\ y_2' &= -d_2p(2,1)o_1y_1, \\ y_3' &= -d_3p(3,1)o_1y_1. \end{aligned} \tag{14}$$

The Lanchester square law provides the answer who can eventually win the modeled battle solely based on the model parameters and initial conditions. It does not provide the solution to the model and thus it does not explain what the precise outcome of the battle is. Furthermore, it does not identify the winner, it only explains who cannot be annihilated. The actual outcome then depends on the length of the battle.

Let $p(i,1) > 0$ then the solution to the equation (14) satisfies the equation

$$y_1^2(t) - \frac{d_1o_2}{d_2o_1p(2,1)}y_2^2(t) - \frac{d_1o_3}{d_3o_1p(3,1)}y_3^2(t) = K, \tag{15}$$

where K is a constant that depends only on the initial conditions $y_1(0)$, $y_2(0)$, $y_3(0)$ and the parameters of the model (14), in other words, on the parameters d_i, o_i and on the function $p(i,1)$.

To prove this law, we need to differentiate (15) with respect to t. This yields

$$2y_1(t)y_1'(t) - \frac{d_1o_2}{d_2o_1p(2,1)}2y_2(t)y_2'(t) - \frac{d_1o_3}{d_3o_1p(3,1)}2y_3(t)y_3'(t) = 0 \tag{16}$$

then substituting (14) into (16) yields that

$$-2y_1(t)d_1\big(o_2y_2(t) + o_3y_3(t)\big) + \frac{d_1o_2}{d_2o_1p(2,1)}2y_2(t)d_2o_1p(2,1)y_1(t) + \frac{d_1o_3}{d_3o_1p(3,1)}2y_3(t)d_3o_1p(3,1)y_1(t) = 0$$

$$-2y_1(t)d_1o_2y_2(t) \pm 2y_1(t)d_1o_3y_3(t) + 2d_1o_2y_2(t)y_1(t) + 2d_1o_3y_3(t)y_1(t) = 0 \tag{17}$$

The left-hand side of equation (17) is equal to zero and the equality holds which asserts that the equality (15) is true.

We claim that if $K > 0$ that the army y_1 is the only one that can eventually win the battle modeled via (14) and on the other hand if $K < 0$ than only armies y_2 and y_3 can eventually win the battle modeled via (14). Here we understand winning as the annihilation of the enemy forces.

The claim follows directly from (16). If we assume that $K > 0$ then y_2 and y_3 win only when y_1 is annihilated and thus eventually must hold $y_1(t) = 0$. If we substitute $y_1(t) = 0$ into (16) then we get

$$-\frac{d_1o_2}{d_2o_1p(2,1)}y_2^2(t) - \frac{d_1o_3}{d_3o_1p(3,1)}y_3^2(t) = K. \tag{18}$$

The left-hand side of equation (18) is negative which is a contradiction with the initial assumption that $K > 0$. Thus $y_1(t) \neq 0$. The other claim if $K < 0$ follows analogously.

6. Discussion

In this paper, we have shown a new MBMFD model that is more realistic than the original multi-battle warfare model. It is vital that MBMFD model enables an incorporation of commander's decision. Figure 2 and 3 depicted a scenario where the multi-battle warfare model expected a loss for the first army. However, this loss originated from an incorrect decision to split the fire equally between enemies. On the other hand, when MBMFD model was used, the commander could focus part of the fire onto the weaker enemy and consequently the first army won, see Figure 3.

It must be acknowledged that even MBMFD model does not incorporate complex tactics that materialize on the battlefield. For example, commander could decide to use complex tactics like flanking or outmaneuvering his opponent. Weather and terrain situation affects spatially the troops' movement and commander's decisions [40–44]. As of now, there appeared attempts to incorporate spatial patterns into the Lanchester battle model [6] but as far as we know they have not been incorporated into multi-battle models which remain as an open problem for the future.

Asymmetric forms of conflict have been considered with respect to a Lanchester battle model in [20] and this can also be incorporated further into multi-battle models. To summarize our claim, there are many further improvements that can be incorporated into multi-battle models in the future which is a work still left to be done for the model's improved realism. We also refer the reader to [45], where the shortcomings of Lanchester's law are summarized.

Our model partially overlaps with the models derived in articles [22,23]. There the authors analyze a battle between three opponents where everyone fights everyone which is distinct from our analysis. Article [46] analyses again such fights between multiple opponents but concludes that there are only two possible outcomes. Either there is an army that can annihilate every other army combined or else all the other armies suffer slow attrition until the eventual stalemate. This outcome is distinct from our own, but we hypothesize that this originates from the fact that we allow cooperation between armies and models in [22,23,46] according to our understanding do not allow cooperation. Furthermore, our model is more general than the models in [22,23,46] that it considers both variable defense and variable offence.

Our work overlaps also with [37] where an army fights against many opponents. Optimal distribution of fire is then derived which describes how the army should split the fire to achieve the best outcome. However, the model in [37] overlaps with ours again because it considers only variable offence but not variable defense. Nevertheless, the results in [37] suggest a path for improvement of our own results because the optimal distribution of fire might be similar in both our model and in the model used in [37]. Nevertheless, we hypothesize that the optimal distribution of fire is logically to always focus on the weakest enemy and then to choose the next weakest enemy after his annihilation.

In a battle between two opponents, Lanchester law enables a simplified analysis of the outcome of the battle because the winner can be inferred the Lanchester law depending on the sign of the right-hand side K. We have derived Lanchester law (15) only for MBMFD model (14). However, even there the analysis was more complex because $y_i(t)$ can become negative at different time points. Therefore, the Lanchester law for multi battle model is weaker than Lanchester law for two opponents. It is unclear as of now if this drawback can be mitigated in the future r even if a Lanchester law for general system (4) can be found.

However, the solutions to the system (4) can be calculated directly for smaller number of participating armies (8). Hence, the winner can be still determined based on the initial conditions and parameters with the sufficiently strong computer algebra system.

In the future, we imagine a potential use case of MBMFD model and its generalizations by a commander who prepares a plan for the battlefield. Individual combat squads can be considered as individual cooperating armies and enemies combat squads can be considered as opposing armies. With sufficiently high computing power and perhaps with the development of artificial intelligence this can lead to the future where software would analyze a conflict in real time and assigns targets to each combat squat separately to gain victory on the battlefield [25,47]. In a way, this would simulate the role of the army officers. Nevertheless, it is unclear as of now if such a future will ever come and this just shows the model's potential.

History always faced the question, what would be the outcome of the battle, had the situation be slightly different, had the commander opted for different tactics, had the soldiers endured a little longer etc. In the past, such questions were only theoretical and led to an academic discussion and the conclusion could not have been verified. Nevertheless, simulation tools enable the verification of such conclusion. This was already done, for example, in [48] where the battle of Gettysburg was analyzed or in [49] where the battle of Thermopylae was modeled. Consequently, the MBMFD model can be of use to historians as well in their analysis of past conflicts and it can be employed for the analysis, for example, of the battle of Austerlitz where Napoleon army faced two different opponent, Russian army and Austrian army.

The argument for the applicability of the MBMFD model could be strengthened in the future by an actual analysis of real conflicts with the actual data. However, realistic conflict data are difficult to obtain, and the model validation thus remains as an open problem for subsequent research.

7. Conclusions

War continues to bring suffering to people around the world. Battle models will not change this fact. But for many people affected by the effects of the war, it would be better if war only took place at the level of simulation and real military conflict could be avoided. Our research provides new and improved simulation tools. If the simulation tools sufficiently progress in the future, we may reach a point where simulations will be so powerful that real conflicts will no longer be necessary.

Our research contributes to the wider analysis of combat models, and it is theoretical in nature. Nevertheless, our results can be applied by historians to analyze past conflicts and likewise by army commanders to analyze ongoing conflicts. In the last century, Lanchester's law attracted attention of many researchers, and we offer only a small contribution to the overall picture. Our aim was not to replace existing knowledge, but rather to refine the existing models and bring forth an example suggesting potential future improvements. In this sense, the value of our work lies less in descriptive conclusions and more in its potential to stimulate further theoretical development and interdisciplinary dialogue.

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